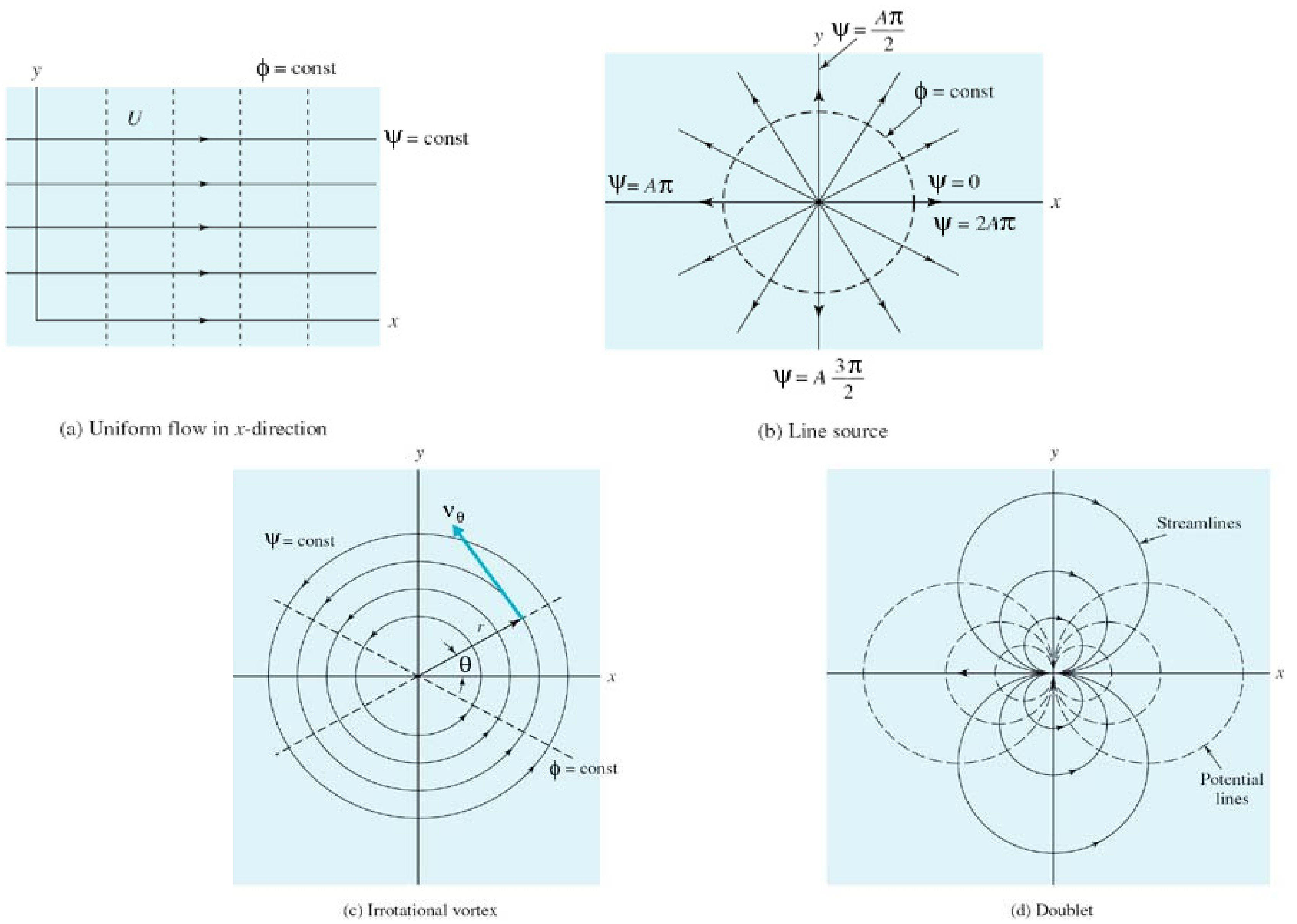
Superposition

Section 8.5.3

Simple Potential Flows

- Most complex potential (invicid, irrotational) flows can be modeled using a combination of simple potential flows
- The simple flows used are:
  - Uniform flows
  - Line sources and sinks
  - Irrotational vortices
  - Doublets
- All of these simple flows satisfy the Laplace equation

Simple Potential Flows



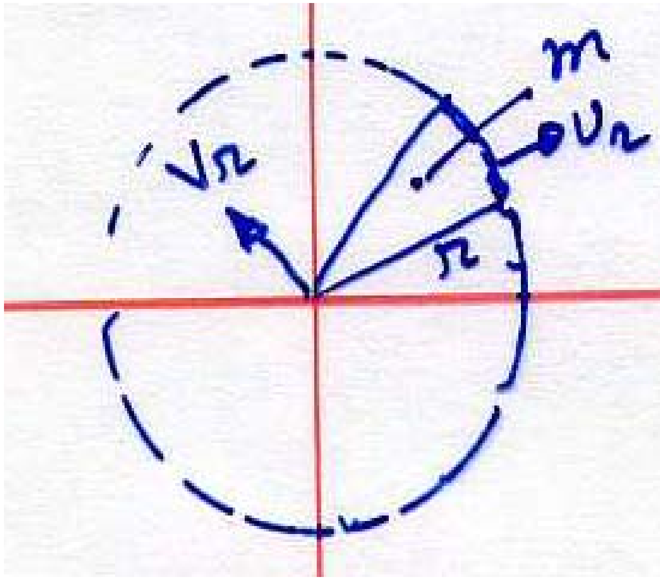
Superposition of potential flows

- If ψ_1 and ψ_2 are two independent solutions of Laplace eqn then $\psi = \psi_1 + \psi_2$ satisfies Laplace eqn.
- So stream function of a complex flow ψ can be found by adding stream functions (ψ_1, ψ_2) of two simpler flows

Note: The above conclusion is also valid for potential functions

Sources & Sinks

- If fluid flows radially outward it is a **source**
- For radially inward it is **sink**.



- Let q be the volume flow rate per unit

length of a line passing through center

As there is no flow in angular direction

$$v_\theta = 0 = -\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

Therefore the volume flow (per unit length) of a source/sink is only in the radial direction:

$2*\pi*r*v_r = q$ [Note q is - ve for Sink] Here q is called Strength

$$v_r = \frac{q}{2\pi r} = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r}$$

$$\boxed{\psi = \frac{q}{2\pi}\theta = m\,\theta}$$

$$\boxed{\phi = \frac{q}{2\pi}\ln r = m \ln r}$$

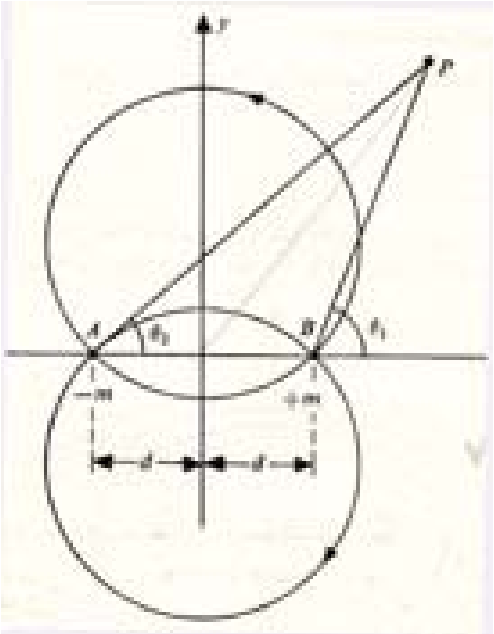
PLANE - FLOW SOLUTIONS

a) Source and Sink of equal strength

- Source, m at $(-d,0)$ and a sink, m at $(d,0)$
- For source $\psi_1 = m\theta_1$. For sink $\psi_2 = -m\theta_2$
  (where, q = strength of source, $m = q/2\pi$)
- For combined flow $\psi = \psi_1 + \psi_2 = m(\theta_1 - \theta_2)$.

$$\psi = m(\theta_1 - \theta_2)$$

$$\phi = \frac{1}{2} m(\ln r_1 - \ln r_2)$$



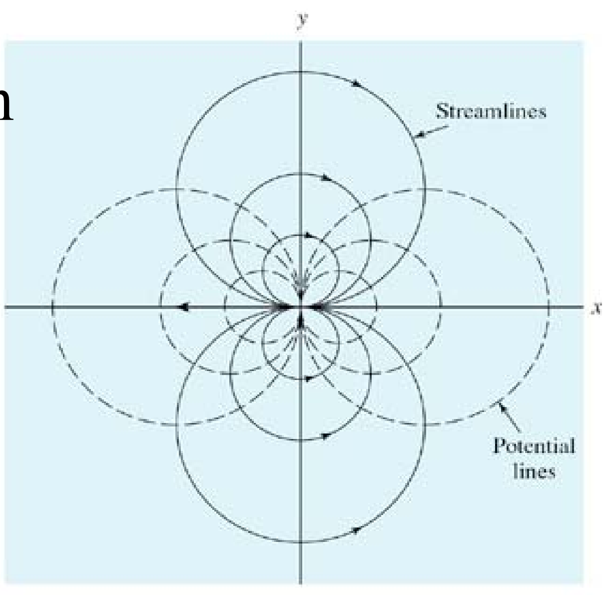
b) Doublet

It is a special case of source and sink when both approach the origin while keeping the product of strength and distance constant ($q*d/2\pi$ = constant = μ.)

It can be proved that

$$\psi = -\frac{\mu \sin\theta}{r}$$

$$\phi = \frac{\mu}{r}\cos\theta$$



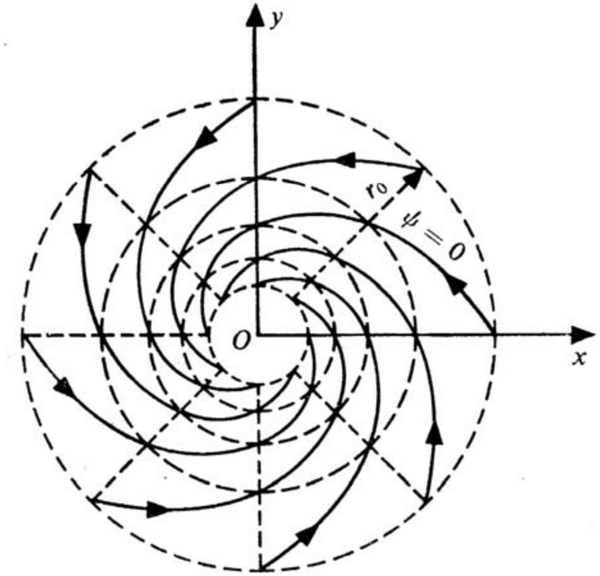
(d) Doublet

Stream & Potential functions of some standard flows

Types of flow	Stream, ψ	Potential ϕ
Uniform flow in x direction	$U\, r \sin\theta = U\, y$	$U\, r \cos\theta = U\, x$
Source	$m\theta$	$m \ln r$
Sink	$- m\theta$	$- m \ln r$
Free vortex (anticlockwise)	$-K \ln r$	$K\,\theta$
Doublet	$-\dfrac{\mu \sin\theta}{r}$	$\dfrac{\mu \cos\theta}{r}$

Where $K = (\Gamma/2\pi)$, $m = q\,/2\pi$; Circulation, Γ can be +ve or –ve.

Superposition of simple flows



SPIRAL VORTEX

(Line sink, m) + (Free vortex anticlockwise, Γ)

$$\psi = \psi_{sink} + \psi_{vortex} = -m\theta - K \ln r$$

Similarly $\phi = -m \ln r + K\theta$

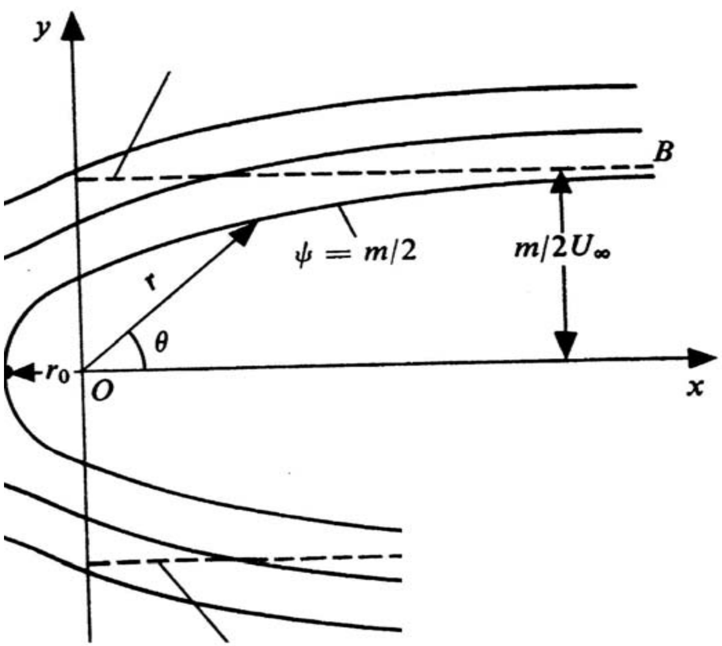
Quiz:How would you form an outward spiral ?

FLOW OVER HALF BODY

(Uniform flow) + (Source)

$$\Psi = U\, r \sin\theta + m\ \theta$$

$$\Phi = U\, r \cos\theta + m \ln r$$



Flow past a cylinder

- This can be simulated by superimposing a doublet on a uniform flow

$$\psi = \psi_{uniform\ \text{flow}} + \psi_{\text{doublet}}$$

$$\Psi = U\, r \sin\theta \quad - \frac{\lambda \sin\theta}{r} \quad = (U.r - \lambda/r)\sin\theta$$

- **Streamline on object's surface is zero**. So for the surface $\sin\theta(U.r - \lambda/r) = 0$;
  or $r = (\lambda/U)^{0.5}$. It is the equation of a circle
- Stagnant points occurs at $V_\theta = 0$;

which occurs at 0 & 180.

$$v_\theta = -\frac{\partial\psi}{\partial r} = (U + \frac{\lambda}{r^2})\sin\theta$$

Velocity on the cylinder surface is found on substitution

$$v_r = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \left(U - \frac{\lambda}{r^2}\right)\cos\theta$$

Drag

- **Drag** is found by integrating force in flow direction and this integral is found to be zero

$$D = -\int_0^{2\pi} P_s (ba) d\theta . \cos\theta = 0$$

It suggests that there is no drag on a cylinder in cross-flow on a stationery cylinder

- This absurd result is called D'Alembert's Paradox.

ROTATING CYLINDER

Flow over rotating cylinder = Uniform flow + Free Vortex (anticlockwise) + Doublet

$$\Psi = U\, r \sin\theta + (-K \ln r) - (\lambda \sin\theta / r)$$

Velocity at a point

$$v_\theta = -\frac{\partial \psi}{\partial r} = -U \sin\theta + \frac{K}{r} - \frac{\lambda \sin\theta}{r^2}$$

Vortex flow does not change V_r . So it remains unchanged.

So, the surface is still defined by $\lambda = U a^2$

Velocity on its surface is therefore = $V_\theta = -2\, U \sin\theta + (K/a)$

Stagnation point occurs where $V_\theta = 0$,

$$\sin\theta_s = \frac{K}{2aU}$$

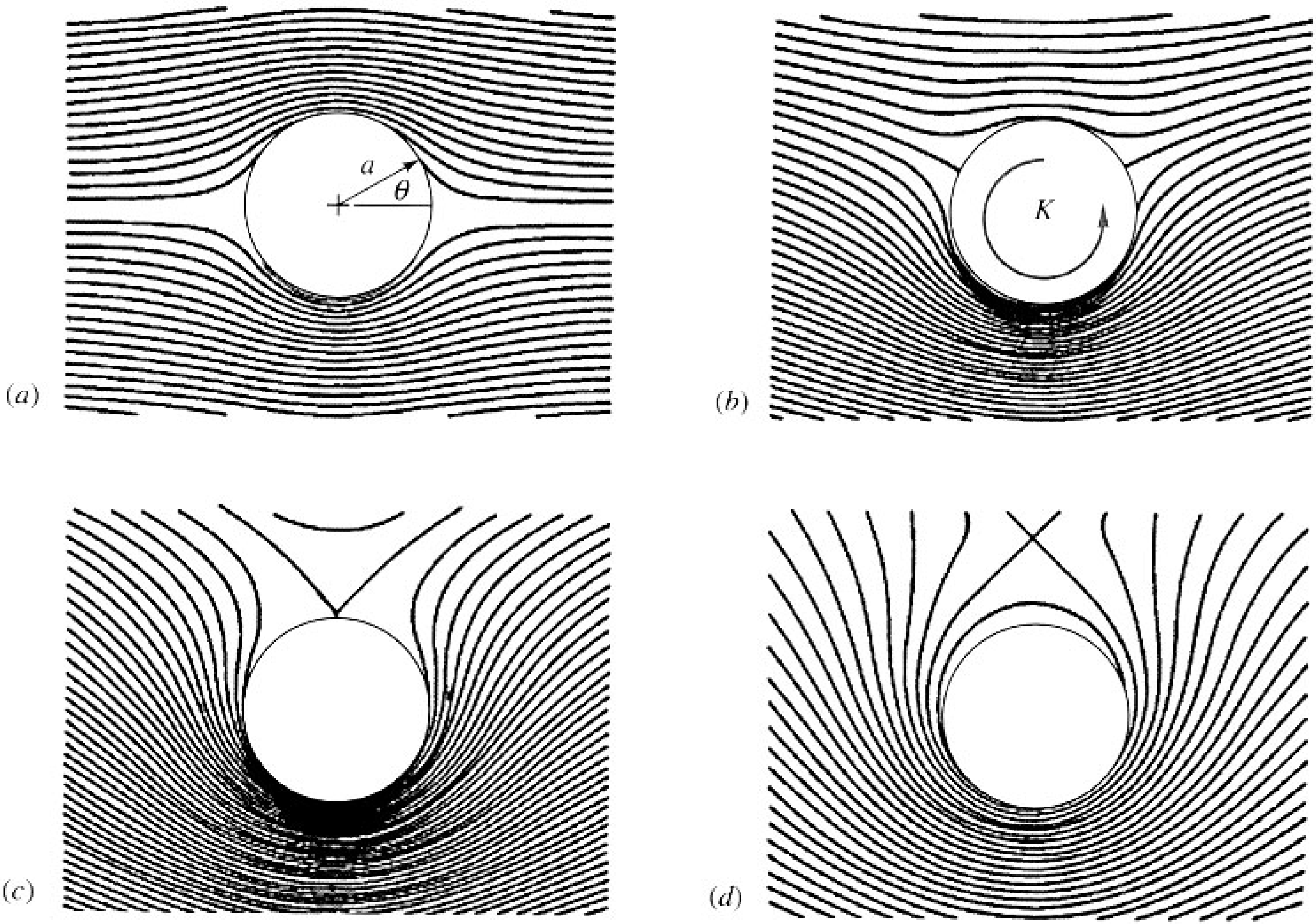
So, no circulation (K=0), stagnation occurs at $\theta_s = 0$ and 180^0

<u>For rotation</u>

If $K = aU$, $\sin\theta = 0.5$; stagnation occurs at $\theta_s = 30$ and 150^0

$\Gamma = 2\,\pi K$ where Γ is circulation

Stream lines around a rotating cylinder



ROTATING CYLINDER

Applying Bernoullis eqn. between the surface (where velocity is v_θ) and freestream we get

$$P_\infty + \frac{1}{2}\rho U^2 = P_s + \frac{1}{2}\rho V_\theta^2 = P_s + \frac{1}{2}\rho\left(-2U\sin\theta + \frac{K}{a}\right)^2$$

$$P_s - P_\infty = \frac{1}{2}\rho U^2\left(1 - 4\sin^2\theta + 4\beta\sin\theta - \beta^2\right) \quad \text{where} \quad \beta = \frac{K}{Ua}$$

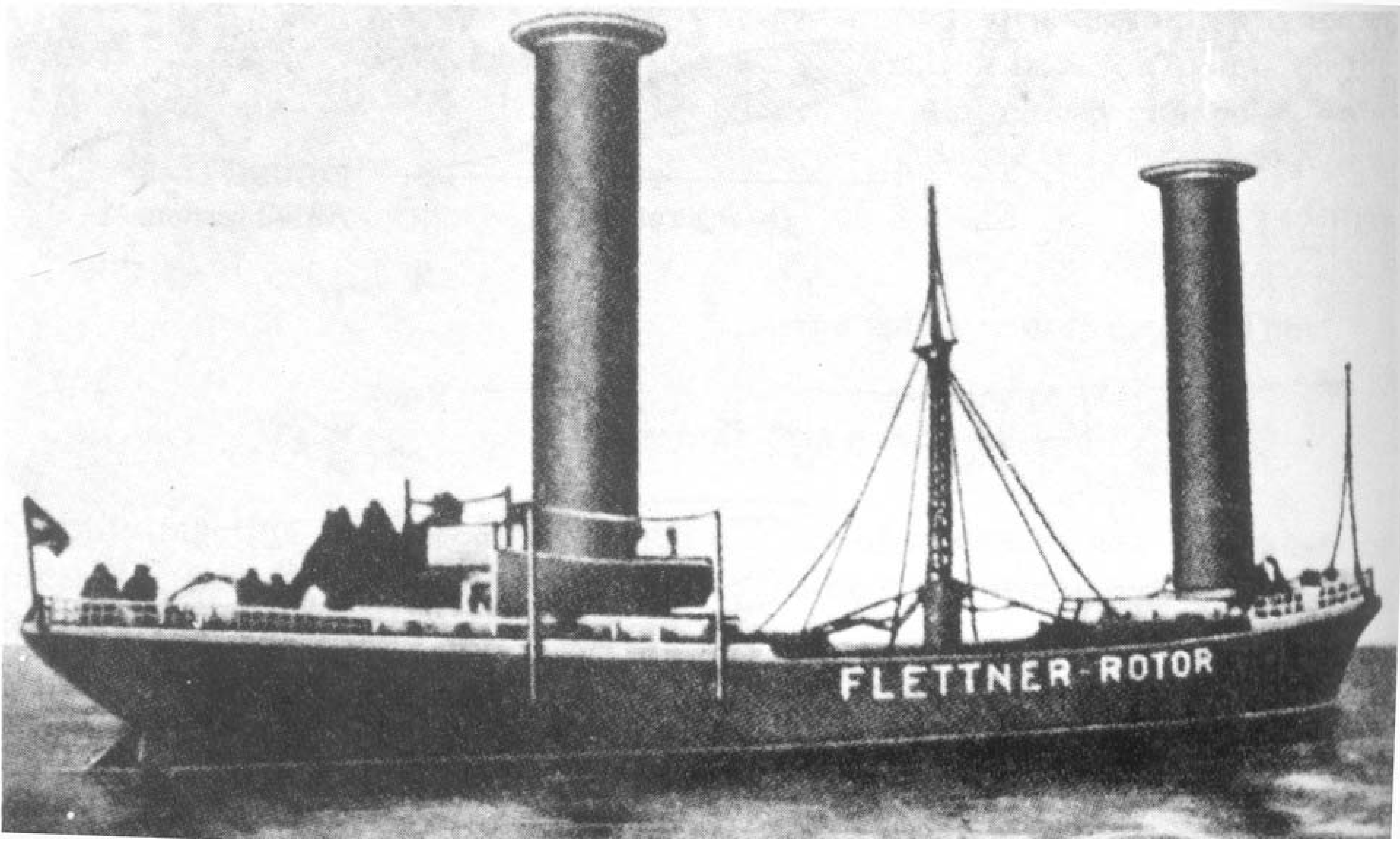
Lift, is found by integrating forces normal to the flow

$$L = -\int_0^{2\pi}\left(P_s - P_\infty\right)\sin\theta\left(ba\right)d\theta = -\rho U\left(2\pi K\right)b = -\rho U\Gamma b$$

$$\frac{L}{b} = -\rho U\Gamma \qquad \text{where a = radius of cylinder \& b = length of cylinder}$$

Upward force, Lift on a rotating cylinder is generated when circulation is negative. It is called the **Magnus effect**

Developed by Mr. Flettner in 1927



Problem - 1

a) Find the stream function (in polar co-ordinate) in an uniform flow in x direction.

b) In a two-dimensional incompressible flow the fluid velocity components are given by $u = x - 4y$ and $v = -y - 4x$. Find the stream function and show that it is an irrotational flow

Problem - 2

- In the ideal flow around a half body, the free stream velocity is 0.5 m/s and the strength of the source is $1/\pi$ m^2/s. Determine the fluid velocity and its direction at a point $r = 1$ m and $\theta = 120$

Problem on Magnus Effect

- In 1927 a man named Flettner had a ship built with two rotating cylinders to act as sails. The height of the cylinders is 15 m and diameter 2.75 m, The wind speed is 30 km/h, and the speed of the ship is 4 km/h. The cylinders rotate at a speed of 750 rpm by the action of a steam engine below deck. Air density 1.229 kg/m3.
- Find the maximum possible propulsive thrust on the ship from the cylinders.
  - [Shames p-597]

Home work (White 4.72)

- A coastal power plant takes in cooling water through a vertical perforated manifold. The total volume flow intake is 110 m^3/s. Currents of 0.25 m/s flow past the manifold. Estimate how far downstream and how far normal to the paper the effects of the intake are felt in the ambient 8 m deep waters. [8.75 m, 55 m]